

REMARKS ON SYMPLECTIC MEAN CURVATURE FLOWS IN KÄHLER SURFACES WITH POSITIVE HOLOMORPHIC SECTIONAL CURVATURES

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ABSTRACT. In this paper, we mainly study the mean curvature flow in Kähler surfaces with positive holomorphic sectional curvatures. First, we prove that if the ratio λ of the maximum and the minimum of the holomorphic sectional curvatures < 2 , then there exists a positive constant $\delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$ such that $\cos \alpha \geq \delta$ is preserved along the flow, improving the main theorem in [LY]; Secondly, as similar as the main theorem in [HL0], we prove that when $\cos \alpha$ is close to 1 enough, then the symplectic mean curvature flow exists for long time and converges to a holomorphic curve; Finally, we prove that the symplectic mean curvature flow on Kähler surfaces with $\lambda \leq 1 + \frac{1}{200}$ exists for long time and converges to a holomorphic curve if the initial surface satisfies a pinching condition, which generalize one of the main theorems in [HLY].

1. INTRODUCTION

Let (M, J, ω, \bar{g}) be a Kähler surface. For a compact oriented real surface Σ which is smoothly immersed in M , the Kähler angle α of Σ in M was defined by

$$\omega|_{\Sigma} = \cos \alpha d\mu_{\Sigma}$$

where $d\mu_{\Sigma}$ is the area element of Σ in the induced metric from g . We say that Σ is a symplectic surface if $\cos \alpha > 0$; Σ is a holomorphic curve if $\cos \alpha = 1$.

It is important to find the conditions to assure that the symplectic property is preserved along the mean curvature flow. In the case that M is a Kähler-Einstein surface, the symplectic property is preserved. If the ambient Kähler surface evolves along the Kähler-Ricci flow, Han and Li [HL1] proved that the symplectic property is also preserved. In [LY], Li and Yang found another condition to assure that the symplectic property is preserved along the mean curvature flow. In this note, we will improve their conditions to assure that along the flow.

In this paper we only consider the ambient Kähler surface with positive holomorphic sectional curvature. Denote the minimum and maximum of holomorphic sectional curvatures of M by k_1 and k_2 , and $\lambda = \frac{k_2}{k_1}$. Then we have the first theorem.

Theorem 1.1. *Suppose M is a Kähler surface with positive holomorphic sectional curvatures. If $1 \leq \lambda < 2$ and $\cos \alpha(\cdot, 0) \geq \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, then along the flow*

$$(1) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha \geq |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha + C \sin^2 \alpha,$$

where C is a positive constant depending only on k_1, k_2 and δ . As a corollary, $\min_{\Sigma_t} \cos \alpha$ is increasing with respect to t . In particular, at each time t , Σ_t is symplectic.

Remark 1.1.1. The main theorem in [LY], the lower bound of δ is $\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}$ for $\lambda \in [1, \frac{11}{7})$, and $\frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$ for $\lambda \in [\frac{11}{7}, 2)$. It is easy to check that for each $\lambda \in [1, 2)$, then

$$\frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}} \leq \min\left\{\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}, \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}\right\}.$$

Hence we improving the result of the Main Theorem in [LY].

In analogy to the main theorem in [HL0], we also prove the following theorem for a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$.

Theorem 1.2. Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos \alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24\lambda)^2+(58\lambda-58)^2}}$. Then there exists a sufficiently small constant ϵ_1 such that if $\int_{\Sigma_0} \frac{\sin^2 \alpha(x,0)}{\cos \alpha(x,0)} d\mu_0 \leq \epsilon_1$ and ϵ_1 satisfying

$$\epsilon_1 \leq \frac{\pi^2 \epsilon_0^2 r_0^6 (1 - e^{-\frac{3}{8}(2-\lambda)k_1})^2}{4 \text{Area}(\Sigma_0)},$$

there r_0 is defined in Remark 4.1.1, and ϵ_0 is the constant in White's theorem (Theorem 4.1 in the present paper), the mean curvature flow with initial surface Σ_0 exists globally and it converges to a holomorphic curve.

As a consequence, we obtain that

Corollary 1.2.1. Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. If there exists a positive constant δ such that

$$\cos \alpha(\cdot, 0) \geq \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$$

and has the longtime existence. Then the mean curvature flow converges to a holomorphic curve.

Han, Li and Yang proved the following theorem with constant holomorphic sectional curvature, see Theorem 3.2, Theorem 4.1 and Theorem 4.2 in [HLY].

Theorem 1.3 (Han-Li-Yang). Suppose Σ is a symplectic surface in CP^2 with constant holomorphic sectional curvature $k > 0$. Assume that $|A|^2 \leq \sigma|H|^2 + \frac{2\sigma-1}{\sigma}k$ and $\cos \alpha \geq \sqrt{\frac{7\sigma-3}{3\sigma}}$ holds on the initial surface for any $\frac{1}{2} < \sigma \leq \frac{2}{3}$, then it remains true along the symplectic mean curvature flow. Furthermore, the symplectic mean curvature flow exists for long time and converges to a holomorphic curve at infinity.

In this paper, we also prove the similar theorem with positive holomorphic sectional curvature.

Theorem 1.4. Suppose Σ is a symplectic surface in the Kähler surface (M, J, ω, \bar{g}) with positive holomorphic sectional curvature and $1 \leq \lambda < 1 + \frac{1}{200}$, and $|\bar{\nabla} Rm| \leq Kk_1(\lambda-1)$ for a positive constant $K \leq \min\{2, 2k_1\}$. For any σ satisfying $\frac{1}{2} + \frac{24(\lambda-1)}{1-34(\lambda-1)} < \sigma \leq \frac{2}{3}$, and we set

$$b = \frac{2\sigma-1}{\sigma}(8-7\lambda) - 4K(\lambda-1),$$

and

$$\begin{aligned}
 a_1 &= 9(\lambda + 1)^2, \\
 a_2 &= 9(\lambda + 1)^2 - \frac{12(3 - 4\sigma)}{2\sigma - 1}b, \\
 a_3 &= \frac{350 - 444\sigma}{2\sigma - 1}(\lambda - 1) + \frac{8(3 - 4\sigma)}{2\sigma - 1}(23\lambda - \frac{41}{2})b - \frac{8(3 - 4\sigma)(\sigma + 1)}{(2\sigma - 1)^2}b^2. \\
 t_0 &= \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_1}, \quad \delta = \max\{t_0, \frac{13\lambda - 10}{3(\lambda + 2)}\}
 \end{aligned}$$

If we assume that $|A|^2 \leq \sigma|H|^2 + bk_1$ and $\cos \alpha \geq \sqrt{\delta}$ holds on the initial surface, then it remains true along the symplectic mean curvature flow. Furthermore, the symplectic flow exists for long time and converges to a holomorphic curve at infinity.

Remark 1.4.1. If $\lambda = 1$, then $\frac{13\lambda - 10}{3(\lambda + 2)} = \frac{1}{3}$ and

$$t_0 = \frac{7\sigma - 3 + \sqrt{(7\sigma - 3)^2 + 4(3 - 4\sigma)(3\sigma - 2)}}{6\sigma} \leq \frac{7\sigma - 3}{3\sigma}.$$

Hence even in the case of $\lambda = 1$, the above theorem improving Han-Li-Yang's Theorem (Theorem 1.3).

Remark 1.4.2. By Lemma 2.3, if $\lambda < \frac{3}{2}$, then the sectional curvature of M is positive, implies that the bisectional curvature is positive. By Frankel conjecture, which was proved by Siu-Yau [SY] using harmonic maps and by Mori [M] via algebraic methods, then the Kähler surface is biholomorphic to \mathbb{CP}^2 .

2. PRELIMINARIES

2.1. Evolution Equations. Suppose that Σ is a sub manifold in a Riemannian manifold M , we choose an orthonormal basis $\{e_i\}$ for $T\Sigma$ and $\{e_\alpha\}$ for $N\Sigma$. Given an immersed $F_0 : \Sigma \rightarrow M$, we consider a one parameter family of smooth maps $F_t = F(\cdot, t) : \Sigma \rightarrow M$ with corresponding images $\Sigma_t = F_t(\Sigma)$ immersed in M and F satisfies the mean curvature flow equation:

$$(2) \quad \begin{cases} \frac{\partial}{\partial t} F(x, t) = H(x, t) \\ F(x, 0) = F_0(x) \end{cases}$$

Recall the evolution equation for the second fundamental form h_{ij}^α and $|A|^2$ along the mean curvature flow (see[CJ1], [HLY], [S1], [W]).

Lemma 2.1.

$$\begin{aligned}
 (3) \quad \frac{\partial}{\partial t} h_{ij}^\alpha &= \Delta h_{ij}^\alpha + (\bar{\nabla}_k Rm)_{\alpha i j k} + (\bar{\nabla}_j Rm)_{\alpha k i k} - 2R_{l i j k} h_{l k}^\alpha \\
 &+ 2R_{\alpha \beta j k} h_{i k}^\beta + 2R_{\alpha \beta i k} h_{j k}^\beta - R_{l k i k} h_{l j}^\alpha - R_{l k j k} h_{i l}^\alpha \\
 &+ R_{\alpha k \beta k} h_{i j}^\beta - H^\beta (h_{i k}^\beta h_{j k}^\alpha + h_{j k}^\beta h_{i k}^\alpha) + h_{i m}^\alpha h_{m k}^\beta h_{k j}^\beta \\
 &- 2h_{i m}^\beta h_{m k}^\alpha h_{k j}^\beta + h_{i k}^\beta h_{k m}^\beta h_{m j}^\alpha + h_{k m}^\alpha h_{m k}^\beta h_{i j}^\beta + h_{i j}^\beta < e_\beta, \bar{\nabla}_H e_\alpha >,
 \end{aligned}$$

where R_{ABCD} is the curvature tensor of M and $\bar{\nabla}$ is the covariant derivative of M . Therefore

$$(4) \quad \begin{aligned} \frac{\partial}{\partial t}|A|^2 = & \Delta A - 2|\nabla A|^2 + [(\bar{\nabla}_k Rm)_{\alpha i j k} + (\bar{\nabla}_j Rm)_{\alpha k i k}]h_{ij}^\alpha \\ & - 4R_{lij k}h_{lk}^\alpha h_{ij}^\alpha + 8R_{\alpha\beta j k}h_{ik}^\beta h_{ij}^\alpha - 4R_{lk i k}h_{lj}^\alpha h_{ij}^\alpha + 2R_{\alpha k \beta k}h_{ij}^\beta h_{ij}^\alpha + 2P_1 + 2P_2 \end{aligned}$$

where

$$\begin{aligned} P_1 = & \Sigma_{\alpha,\beta,i,j}(\Sigma_k(h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta))^2, \\ P_2 = & \Sigma_{\alpha,\beta}(\Sigma_{i,j}h_{ij}^\alpha h_{ij}^\beta)^2. \end{aligned}$$

$$(5) \quad \frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2R_{\alpha k \beta k}H^\alpha H^\beta + 2P_3$$

where

$$P_3 = \Sigma_{i,j}(\Sigma_\alpha H^\alpha h_{ij}^\alpha)^2.$$

Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on (M, \bar{g}) along Σ_t such that $\{e_1, e_2\}$ is the frame of the tangent bundle $T\Sigma_t$ and $\{e_3, e_4\}$ is the frame of the normal bundle $N\Sigma_t$. Then along the surface Σ_t , we can take the complex structure on M as the form (cf. [HLY])

$$(6) \quad J = \begin{pmatrix} 0 & \cos \alpha & y & z \\ -\cos \alpha & 0 & -z & y \\ -y & z & 0 & -\cos \alpha \\ -z & -y & \cos \alpha & 0 \end{pmatrix}$$

or

$$(7) \quad J = \begin{pmatrix} 0 & \cos \alpha & y & z \\ -\cos \alpha & 0 & z & -y \\ -y & -z & 0 & \cos \alpha \\ -z & y & -\cos \alpha & 0 \end{pmatrix}$$

Since Kähler form is self-dual, then J must be the form (7).

Remark 2.0.3. In fact, the above argument also shows that the Kähler form is self-dual. If J is the form (6), then the Kähler form is anti-self-dual, i.e., $*\omega = -\omega$, it is impossible for Kähler form. Hence J must be the form (7), then the Kähler form ω must be self-dual.

Recall the evolution equation of the Kähler angle $\cos \alpha$ (cf. [CL1], [HL2]),

Lemma 2.2. The evolution equation for $\cos \alpha$ along Σ_t is

$$(8) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha = |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha + \sin^2 \alpha \operatorname{Ric}(Je_1, e_2).$$

Here

$$(9) \quad |\bar{\nabla} J_{\Sigma_t}|^2 = |h_{1k}^4 + h_{2k}^3|^2 + |h_{2k}^4 - h_{1k}^3|^2.$$

$|\bar{\nabla} J_{\Sigma_t}|^2$ is independent of the choice of the frame and only depend on the orientation of the frame. It is proved in ([CL1], [HL1]) that

$$(10) \quad |\bar{\nabla} J_{\Sigma_t}|^2 \geq \frac{1}{2}|H|^2$$

and

$$(11) \quad |\nabla \cos \alpha|^2 \leq \sin^2 \alpha |\bar{\nabla} J_{\Sigma_t}|^2.$$

2.2. Curvatures. In this subsection, we first recall the definitions of Riemannian curvature and the holomorphic sectional curvature; secondly, we give some estimates of Riemannian curvatures by holomorphic sectional curvature.

The Riemann curvature tensor R of (M, g) is defined by

$$R(X, Y, Z, W) = -g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W)$$

for any vector fields X, Y, Z, W .

Set $R(X, Y) = R(X, Y, X, Y)$ and $R(X) = R(X, JX)$. Fix a point $p \in M$ and a two-dimensional plane $\Pi \subset T_p M$. The sectional curvature of Π is defined by

$$K(\Pi) = \frac{R(X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}$$

where $\{X, Y\}$ is a basis of Π , we also denote it by $K(X, Y)$. For a Kähler manifold (M, g, J) , if the two-dimensional plane Π is spanned by $\{X, JX\}$, i.e., Π is a holomorphic plane, then the sectional curvature of Π is called a holomorphic sectional curvature of Π , we denote it by $K(X)$, where $\{X, JX\}$ is a basis of Π . Then

$$K(X) = \frac{R(X)}{g(X, X)^2}.$$

It is well known that we can express the sectional curvatures by holomorphic sectional curvatures.

Theorem 2.1. *The sectional curvatures of M can be determined by holomorphic sectional curvatures by*

$$(12) \quad R(X, Y) = \frac{1}{32} [3R(X + JY) + 3R(X - JY) - R(X + Y) - R(X - Y) - 4R(X) - 4R(Y)].$$

Theorem 2.2. *For any vector fields X, Y and Z on M ,*

$$(13) \quad R(X, Y, X, Z) = \frac{1}{4} (R(X, Y + Z) - R(X, Y - Z)).$$

Lemma 2.3. *For any two orthogonal vectors X and Y , set $|X|^2 = a, |Y|^2 = b, \langle JX, Y \rangle = x$, then*

$$(14) \quad R(X, Y) \leq \frac{1}{16} [(3(a + b)^2 + 12x^2)k_2 - (3a^2 + 3b^2 + 2ab)k_1]$$

and

$$(15) \quad R(X, Y) \geq \frac{1}{16} [(3(a + b)^2 + 12x^2)k_1 - (3a^2 + 3b^2 + 2ab)k_2]$$

Proof. Since

$$(16) \quad \langle X + JY, X + JY \rangle = |X|^2 - 2\langle JX, Y \rangle + |Y|^2 = a + b - 2x,$$

and

$$(17) \quad \langle X - JY, X - JY \rangle = a + b + 2x, \langle X + Y, X + Y \rangle = \langle X - Y, X - Y \rangle = a + b,$$

by (12), we have

$$\begin{aligned}
 & R(X, Y) \\
 (18) \leq & \frac{1}{32}[3(a+b-2x)^2k_2 + 3(a+b+2x)^2k_2 - (a+b)^2k_1 - (a+b)^2k_1 - 4a^2k_1 - 4b^2k_1] \\
 & = \frac{1}{16}[(3(a+b)^2 + 12x^2)k_2 - (3a^2 + 3b^2 + 2ab)k_1]
 \end{aligned}$$

and similarly

$$(19) \quad R(X, Y) \geq \frac{1}{16}[(3(a+b)^2 + 12x^2)k_1 - (3a^2 + 3b^2 + 2ab)k_2]$$

□

Lemma 2.4. *For the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on (M, g) along Σ_t and takes the form J as (7). Then we have the following estimates:*

- 1) $\frac{1}{4}[(3 + 3\cos^2\alpha)k_1 - 2k_2] \leq R_{1212} \leq \frac{1}{4}[(3 + 3\cos^2\alpha)k_2 - 2k_1];$
- 2) $\frac{1}{4}[(3 + 3\cos^2\alpha)k_1 - 2k_2] \leq R_{3434} \leq \frac{1}{4}[(3 + 3\cos^2\alpha)k_2 - 2k_1];$
- 3) $\frac{1}{4}[(3 + 3y^2)k_1 - 2k_2] \leq R_{1313} \leq \frac{1}{4}[(3 + 3y^2)k_2 - 2k_1];$
- 4) $\frac{1}{4}[(3 + 3y^2)k_1 - 2k_2] \leq R_{2424} \leq \frac{1}{4}[(3 + 3y^2)k_2 - 2k_1];$
- 5) $\frac{1}{4}[(3 + 3z^2)k_1 - 2k_2] \leq R_{1414} \leq \frac{1}{4}[(3 + 3z^2)k_2 - 2k_1];$
- 6) $\frac{1}{4}[(3 + 3z^2)k_1 - 2k_2] \leq R_{2323} \leq \frac{1}{4}[(3 + 3z^2)k_2 - 2k_1];$
- 7) $\frac{1}{32}[(23 + 6(\cos\alpha + y)^2)k_1 - (23 + 6(\cos\alpha - y)^2)k_2] \leq R_{2131} \leq \frac{1}{32}[(23 + 6(\cos\alpha + y)^2)k_2 - (23 + 6(\cos\alpha - y)^2)k_1];$
- 8) $\frac{1}{32}[(23 + 6(\cos\alpha - y)^2)k_1 - (23 + 6(\cos\alpha + y)^2)k_2] \leq R_{2434} \leq \frac{1}{32}[(23 + 6(\cos\alpha - y)^2)k_2 - (23 + 6(\cos\alpha + y)^2)k_1];$
- 9) $\frac{1}{32}[(23 + 6(\cos\alpha + y)^2)k_1 - (23 + 6(\cos\alpha - y)^2)k_2] \leq R_{1242} \leq \frac{1}{32}[(23 + 6(\cos\alpha + y)^2)k_2 - (23 + 6(\cos\alpha - y)^2)k_1];$
- 10) $\frac{1}{32}[(23 + 6(\cos\alpha - z)^2)k_1 - (23 + 6(\cos\alpha + z)^2)k_2] \leq R_{1232} \leq \frac{1}{32}[(23 + 6(\cos\alpha - z)^2)k_2 - (23 + 6(\cos\alpha + z)^2)k_1];$
- 11) $\frac{1}{32}[(23 + 6(\cos\alpha + z)^2)k_1 - (23 + 6(\cos\alpha - z)^2)k_2] \leq R_{2141} \leq \frac{1}{32}[(23 + 6(\cos\alpha + z)^2)k_2 - (23 + 6(\cos\alpha - z)^2)k_1];$
- 12) $\frac{1}{32}[(23 + 6(y+z)^2)k_1 - (23 + 6(y-z)^2)k_2] \leq R_{3141} \leq \frac{1}{32}[(23 + 6(y+z)^2)k_2 - (23 + 6(y-z)^2)k_1];$
- 13) $\frac{1}{32}[(23 + 6(y-z)^2)k_1 - (23 + 6(y+z)^2)k_2] \leq R_{3242} \leq \frac{1}{32}[(23 + 6(y-z)^2)k_2 - (23 + 6(y+z)^2)k_1];$
- 14) $\frac{1}{12}[(10 + 6\cos^2\alpha)k_1 - (10 + 3\sin^2\alpha)k_2] \leq R_{1234} \leq \frac{1}{12}[(10 + 6\cos^2\alpha)k_2 - (10 + 3\sin^2\alpha)k_1].$
- 15) $\frac{1}{2}(6k_1 - 3k_2) \leq R_{ii} \leq \frac{1}{2}(6k_2 - 3k_1) (1 \leq i \leq 4).$
- 16) $|R_{34}| \leq \frac{29-6\cos^2\alpha}{16}(k_2 - k_1).$

Proof. By (7), we have

- $Je_1 = \cos\alpha e_2 + ye_3 + ze_4,$
- $Je_2 = -\cos\alpha e_1 + ze_3 - ye_4,$
- $Je_3 = -ye_1 - ze_2 + \cos\alpha e_4,$
- $Je_4 = -ze_1 + ye_2 - \cos\alpha e_3.$

Hence $\langle Je_1, e_2 \rangle = \langle Je_3, e_4 \rangle = \cos\alpha$, $\langle Je_1, e_3 \rangle = \langle Je_4, e_2 \rangle = y$, $\langle Je_1, e_4 \rangle = \langle Je_2, e_3 \rangle = z$, then by Lemma 2.3, we get 1)-6).

By Theorem 2.2,

$$(20) \quad R_{1213} = \frac{1}{4}(R(e_1, e_2 + e_3) - R(e_1, e_2 - e_3)).$$

Since $Je_1 = \cos \alpha e_2 + ye_3 + ze_4$, $\langle Je_1, e_2 + e_3 \rangle = \cos \alpha + y$ and $\langle Je_1, e_2 - e_3 \rangle = \cos \alpha - y$. Then by Lemma 2.3,

$$(21) \quad \frac{1}{16}[(27 + 12(\cos \alpha + y)^2)k_1 - 19k_2] \leq R(e_1, e_2 + e_3) \leq \frac{1}{16}[(27 + 12(\cos \alpha + y)^2)k_2 - 19k_1],$$

and

$$(22) \quad \frac{1}{16}[(27 + 12(\cos \alpha - y)^2)k_1 - 19k_2] \leq R(e_1, e_2 - e_3) \leq \frac{1}{16}[(27 + 12(\cos \alpha - y)^2)k_2 - 19k_1],$$

Hence

$$(23) \quad \begin{aligned} R_{1213} &\leq \frac{1}{64}[(46 + 12(\cos \alpha + y)^2)k_2 - (46 + 12(\cos \alpha - y)^2)k_1] \\ &= \frac{1}{32}[(23 + 6(\cos \alpha + y)^2)k_2 - (23 + 6(\cos \alpha - y)^2)k_1] \end{aligned}$$

and

$$(24) \quad R_{1213} \geq \frac{1}{32}[(23 + 6(\cos \alpha + y)^2)k_1 - (23 + 6(\cos \alpha - y)^2)k_2]$$

Hence we obtain 7).

Using Theorem 2.2, Lemma 2.3 and the same argument as in the proof of 7), we can obtain 8)-13).

It is easy to check the following identity

$$(25) \quad \begin{aligned} 24R_{1234} &= R(e_1 + e_3, e_2 + e_4) - R(e_1 + e_3, e_2 - e_4) - R(e_1 - e_3, e_2 + e_4) + R(e_1 - e_3, e_2 - e_4) \\ &\quad - R(e_1 + e_4, e_2 + e_3) + R(e_1 + e_4, e_2 - e_3) + R(e_1 - e_4, e_2 + e_3) - R(e_1 - e_4, e_2 - e_3) \end{aligned}$$

Then by Lemma 2.3, we get

$$(26) \quad 2[(10 + 6\cos^2 \alpha)k_1 - (10 + 3\sin^2 \alpha)k_2] \leq 24R_{1234} \leq 2[(10 + 6\cos^2 \alpha)k_2 - (10 + 3\sin^2 \alpha)k_1].$$

Hence

$$(27) \quad \frac{1}{12}[(10 + 6\cos^2 \alpha)k_1 - (10 + 3\sin^2 \alpha)k_2] \leq R_{1234} \leq \frac{1}{12}[(10 + 6\cos^2 \alpha)k_2 - (10 + 3\sin^2 \alpha)k_1]$$

Therefore we get 14).

By 1)-14), it is easy to get 15) and 16). \square

3. LOWER BOUND ALONG THE MEAN CURVATURE FLOW

In this section, we following the argument in [LY] to prove the first main theorem of this paper, which improving the main theorem in [LY].

Theorem 3.1. *Suppose M is a Kähler surface with positive holomorphic sectional curvatures. If $1 \leq \lambda < 2$ and $\cos \alpha(\cdot, 0) \geq \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2 + (29\lambda-29)^2}}$, then along the flow*

$$(28) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha \geq |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha + C \sin^2 \alpha,$$

where C is a positive constant depending only on k_1, k_2 and δ . As a corollary, $\min_{\Sigma_t} \cos \alpha$ is increasing with respect to t . In particular, at each time t , Σ_t is symplectic.

Proof. For simplicity, we can take $y = \sin \alpha, z = 0$ in the form of J . In order to prove this theorem, we need to estimate $\text{Ric}(Je_1, e_2)$. then

$$\begin{aligned}
 \text{Ric}(Je_1, e_2) &= \sum_{i=1}^4 R(Je_1, e_i, e_2, e_i) \\
 (29) \quad &= \sum_{i=1}^4 R(\cos \alpha e_2 + \sin \alpha e_3, e_i, e_2, e_i) \\
 &= \cos \alpha R_{22} + \sin \alpha (R_{1213} + R_{4243}) \\
 &= \cos \alpha R_{22} + \sin \alpha R_{23}.
 \end{aligned}$$

By Lemma 2.4, we have

$$(30) \quad R_{22} = R_{1212} + R_{3232} + R_{4242} \geq 3k_1 - \frac{3}{2}k_2,$$

and

$$(31) \quad |R_{23}| = |R_{1213} + R_{4243}| \leq \frac{29}{16}(k_2 - k_1).$$

Hence we have

$$\begin{aligned}
 \text{Ric}(Je_1, e_2) &\geq \cos \alpha (3k_1 - \frac{3}{2}k_2) - \sqrt{1 - \cos^2 \alpha} \frac{29}{16}(k_2 - k_1) \\
 (32) \quad &= (3 \cos \alpha + \frac{29}{16} \sqrt{1 - \cos^2 \alpha}) k_1 - (\frac{3}{2} \cos \alpha + \frac{29}{16} \sqrt{1 - \cos^2 \alpha}) k_2.
 \end{aligned}$$

It follows that if $1 \leq \lambda < 2$ and $\cos \alpha > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2 + (29\lambda-29)^2}}$, then the right hand side of (28) is positive. Hence we obtain the Theorem 4.1. \square

Remark 3.1.1. Comparing to the proof in [LY], we do more carefully to estimate the term $R_{1213} + R_{4243}$ by Lemma 2.4.

As same as Corollary 1.2 and Theorem 1.3 in [LY], we also have the following Corollary and Theorem.

Arguing as in [CW] by strong maximum principle, we have

Corollary 3.1.1. Suppose M is a Kähler surface with positive holomorphic sectional curvatures and $1 \leq \lambda < 2$, then every symplectic minimal surface satisfying

$$\cos \alpha > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2 + (29\lambda-29)^2}}$$

in M is a holomorphic curve.

Arguing exactly in the same way as in [CL1] or [W], we have

Theorem 3.2. Under the same condition of Theorem 3.1, then the symplectic mean curvature flow has no type I singularity at any $T > 0$.

4. WHEN $\cos \alpha$ IS CLOSE TO 1

In this section, we will use the same argument of Han and Li [HL0], to prove that Kähler manifold M with positive holomorphic sectional curvature and $1 \leq \lambda < 2$, when $\cos \alpha$ is close enough to 1, then the mean curvature flow exists globally and converges to a holomorphic curve.

Proposition 4.1. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos \alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$. Then*

$$(33) \quad \begin{aligned} \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t &\leq C_0 e^{-\frac{3}{4}(2-\lambda)k_1 t}, \\ \int_t^{t+1} \int_{\Sigma_t} |H|^2 d\mu_t dt &\leq C_0 e^{-\frac{3}{4}(2-\lambda)k_1 t}, \end{aligned}$$

where C_0 is a constant which depends only on the initial surface, $C_0 = \int_{\Sigma_0} \frac{\sin^2 \alpha(x,0)}{\cos \alpha(x,0)} d\mu_0$.

Proof. By Theorem 3.1, we know $\cos \alpha(\cdot, t) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$ is preserved along the mean curvature flow. Since $\cos \alpha > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$, then by (32), we have

$$\text{Ric}(Je_1, e_2) > \frac{3}{4}(2-\lambda)k_1 \cos \alpha.$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha > |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha + \frac{3}{4}(2-\lambda)k_1 \cos \alpha \sin^2 \alpha \geq \frac{3}{4}(2-\lambda)k_1 \cos \alpha \sin^2 \alpha.$$

From the evolution equation of $\cos \alpha$, we have

$$(34) \quad \begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right) \frac{1}{\cos \alpha} \\ &= -\frac{\left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha}{\cos^2 \alpha} - \frac{2|\nabla \cos \alpha|^2}{\cos^3 \alpha} \\ &\leq -\frac{3(2-\lambda)k_1 \sin^2 \alpha}{4 \cos \alpha} - \frac{2|\nabla \cos \alpha|^2}{\cos^3 \alpha} \end{aligned}$$

From the proof of Proposition 2.1 in [HL0], we have $\int_{\Sigma_t} \cos \alpha d\mu_t = \int_{\Sigma_t} \omega$ is constant under the continuous deformation in t .

We therefore have

$$(35) \quad \begin{aligned} &\frac{\partial}{\partial t} \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \\ &= \frac{\partial}{\partial t} \int_{\Sigma_t} \left(\frac{1}{\cos \alpha} - \cos \alpha\right) d\mu_t = \frac{\partial}{\partial t} \int_{\Sigma_t} \frac{1}{\cos \alpha} d\mu_t \\ &= \int_{\Sigma_t} \left(\frac{\partial}{\partial t} - \Delta\right) \frac{1}{\cos \alpha} d\mu_t - \int_{\Sigma_t} \frac{|H|^2}{\cos \alpha} d\mu_t \\ &\leq -\frac{3}{4}(2-\lambda)k_1 \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t - \int_{\Sigma_t} \frac{|H|^2}{\cos \alpha} d\mu_t. \end{aligned}$$

So

$$\frac{\partial}{\partial t} \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \leq -\frac{3}{4}(2-\lambda)k_1 \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t.$$

Then it is easy to get that

$$\int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \leq e^{-\frac{3}{4}(2-\lambda)k_1 t} \int_{\Sigma_0} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_0,$$

that is,

$$\int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \leq C_0 e^{-\frac{3}{4}(2-\lambda)k_1 t}.$$

It follows from (35) that

$$\int_{\Sigma_t} |H|^2 d\mu_t \leq -\frac{\partial}{\partial t} \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t.$$

Integrating the above inequality from t to $t+1$, we obtain that

$$\int_t^{t+1} \int_{\Sigma_t} |H|^2 d\mu_t dt \leq \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \leq C_0 e^{-\frac{3}{4}(2-\lambda)k_1 t}.$$

This proves the proposition. \square

We derive an L^1 -estimation of the mean curvature vector on the time space.

Proposition 4.2. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos \alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2 + (58\lambda-58)^2}}$. Then*

$$(36) \quad \int_0^T \int_{\Sigma_t} |H| d\mu_t dt \leq (C_0)^{1/2} \frac{Area(\Sigma_0)^{1/2}}{1 - e^{-\frac{3}{8}(2-\lambda)k_1}},$$

where the constant C_0 is defined in Proposition 4.1.

Proof. We have

$$\begin{aligned} \int_0^T \int_{\Sigma_t} |H| d\mu_t dt &\leq \sum_{k=0}^{[T]} \int_k^{k+1} \int_{\Sigma_t} |H| d\mu_t dt \\ &\leq \sum_{k=0}^{[T]} \left(\int_k^{k+1} \int_{\Sigma_t} |H|^2 d\mu_t dt \right)^{1/2} \left(\int_k^{k+1} Area(\Sigma_t) \right)^{1/2} \\ &\leq Area(\Sigma_0)^{1/2} \sum_{k=0}^{[T]} \left(\int_k^{k+1} \int_{\Sigma_t} |H|^2 d\mu_t dt \right)^{1/2} \\ &\leq C_0^{1/2} Area(\Sigma_0)^{1/2} \sum_{k=0}^{[T]} e^{-\frac{3}{8}(2-\lambda)k_1 k} \\ &\leq (C_0)^{1/2} \frac{Area(\Sigma_0)^{1/2}}{1 - e^{-\frac{3}{8}(2-\lambda)k_1}}. \end{aligned}$$

This proves the proposition. \square

Remark 4.0.1. Han and Li [HL0] proved the above propositions in the case of Kähler-Einstein manifold M with positive scalar curvature R .

$$\begin{aligned} \int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t &\leq C_0 e^{-Rt}, \\ \int_t^{t+1} \int_{\Sigma_t} |H|^2 d\mu_t dt &\leq C_0 e^{-Rt}, \\ \int_0^T \int_{\Sigma_t} |H| d\mu_t dt &\leq (C_0)^{1/2} \frac{\text{Area}(\Sigma_0)^{1/2}}{1 - e^{-R/2}} \end{aligned}$$

We recall White's local regularity theorem.

Let $H(X, X_0, t)$ be the backward heat kernel on \mathbb{R}^4 . Define

$$(37) \quad \rho(X, t) = 4\pi(t_0 - t)H(X, X_0, t) = \frac{1}{4\pi(t_0 - t)} \exp\left(-\frac{|X - X_0|^2}{4(t_0 - t)}\right)$$

for $t < t_0$. Let i_M be the injective radius of M^4 . We choose a cutoff function $\phi \in C_0^\infty(B_{2r}(X_0))$ with $\phi \equiv 1$ in $B_r(X_0)$, where $X_0 \in M, 0 < 2r < i_M$. Choose normal coordinates in $B_{2r}(X_0)$ and express F using the coordinates (F^1, F^2, F^3, F^4) as a surface in \mathbb{R}^4 . The parabolic density of the mean curvature flow is defined by

$$(38) \quad \Phi(X_0, t_0, t) = \int_{\Sigma_t} \phi(F) \rho(F, t) d\mu_t.$$

The following local regularity theorem was proved by White (see [Wh, Theorems 3.1 and 4.1]).

Theorem 4.1. *There is a positive constant $\epsilon_0 > 0$ such that if*

$$(39) \quad \Phi(X_0, t_0, t_0 - r^2) \leq 1 + \epsilon_0,$$

then the second fundamental form $A(t)$ of Σ_t in M is bounded in $B_{r/2}(X_0)$, that is,

$$(40) \quad \sup_{B_{r/2} \times (t_0 - r^2/4, t_0]} |A| \leq C,$$

where C is a positive constant depending only on M .

Remark 4.1.1. *Since Σ_0 is smooth, it is well known that*

$$\lim_{r \rightarrow 0} \int_{\Sigma_0} \phi(F) \frac{e^{-(|F - X_0|^2/4r^2)}}{4\pi r^2} d\mu_0 = 1$$

for any $X_0 \in \Sigma_0$. So we can find a sufficiently small r_0 such that

$$\int_{\Sigma_0} \phi(F) \frac{e^{-(|F - X_0|^2/4r_0^2)}}{4\pi r_0^2} d\mu_0 \leq 1 + \frac{\epsilon_0}{2}$$

i.e.,

$$\Phi(X_0, r_0^2, 0) \leq 1 + \frac{\epsilon_0}{2}$$

for all $X_0 \in M$, where ϵ_0 is the constant in White's theorem.

We state the main theorem in this section as following.

Theorem 4.2. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos \alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2 + (58\lambda-58)^2}}$. Then there exists a sufficiently small constant ϵ_1 such that if $C_0 \leq \epsilon_1$ and ϵ_1 satisfying*

$$\epsilon_1 \leq \frac{\pi^2 \epsilon_0^2 r_0^6 (1 - e^{-\frac{3}{8}(2-\lambda)k_1})^2}{4 \text{Area}(\Sigma_0)},$$

there C_0 is defined in Proposition 4.1, r_0 is defined in Remark 4.1.1, and ϵ_0 is the constant in White's theorem, the mean curvature flow with initial surface Σ_0 exists globally and it converges to a holomorphic curve.

Proof. The argument is same as in Han and Li [HL0]. For convenience of readers, we provide the argument here.

Fix any positive number T . By the definition of Φ , we have

$$(41) \quad \Phi(X_0, t, t - r^2) = \int_{\Sigma_{t-r^2}} \Phi(F) \frac{e^{-|F-X_0|^2/4r^2}}{4\pi r^2} d\mu_{t-r^2}.$$

Differentiating the above equation with respect to t , we get that

$$\begin{aligned} \frac{\partial}{\partial t} \Phi(X_0, t, t - r^2) &= \int_{\Sigma_{t-r^2}} \langle \nabla \phi, H \rangle \frac{e^{-|F-X_0|^2/4r^2}}{4\pi r^2} d\mu_{t-r^2} \\ &\quad - \int_{\Sigma_{t-r^2}} \frac{\Phi \langle F - X_0, H \rangle}{8\pi r^4} e^{-|F-X_0|^2/4r^2} d\mu_{t-r^2} \\ &\quad - \int_{\Sigma_{t-r^2}} \Phi(F) |H|^2 \frac{e^{-|F-X_0|^2/4r^2}}{4\pi r^2} d\mu_{t-r^2}. \end{aligned}$$

Integrating the above equation from r^2 to T , and set $r = r_0$, then we get

$$\begin{aligned} \Phi(X_0, T, T - r_0^2) &\leq \Phi(X_0, r_0^2, 0) + \int_{r_0^2}^T \int_{\Sigma_{t-r_0^2}} |\nabla \phi| |H| \frac{e^{-|F-X_0|^2/4r_0^2}}{4\pi r_0^2} d\mu_{t-r_0^2} \\ &\quad + \int_{r_0^2}^T \int_{\Sigma_{t-r_0^2}} \frac{\Phi |F - X_0| |H|}{8\pi r_0^4} e^{-|F-X_0|^2/4r_0^2} d\mu_{t-r_0^2}. \end{aligned}$$

Using Remark 4.1.1, Proposition 4.1 and 4.2 and noting that we can choose ϕ such that $|\nabla \phi| \leq \frac{2}{r_0}$, we obtain that

$$\begin{aligned} &\Phi(X_0, T, T - r_0^2) \\ &\leq 1 + \frac{\epsilon_0}{2} + \frac{1}{2\pi r_0^3} (C_0)^{1/2} \frac{\text{Area}(\Sigma_0)^{1/2}}{1 - e^{-\frac{3}{8}(2-\lambda)k_1}} + \frac{1}{2\pi r_0^3} (C_0)^{1/2} \frac{\text{Area}(\Sigma_0)^{1/2}}{1 - e^{-\frac{3}{8}(2-\lambda)k_1}} \\ &\leq 1 + \epsilon_0. \end{aligned}$$

we have use that $e^{-x^2} x \leq e^{-1/2} \frac{\sqrt{2}}{2} < \frac{1}{2}$ in the last inequality.

Applying White's theorem we obtain a uniform estimate of the second fundamental form which implies the global existence and the convergence of the mean curvature flow. By Proposition 4.2, we have that

$$\int_{\Sigma_t} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_t \leq C_0 e^{-\frac{3}{4}(2-\lambda)k_1 t}.$$

Let $t \rightarrow \infty$ and $(2 - \lambda)k_1 > 0$, we get

$$\int_{\Sigma_\infty} \frac{\sin^2 \alpha}{\cos \alpha} d\mu_\infty = 0.$$

It follows that $\cos \alpha_\infty = 1$, that is, Σ_∞ is a holomorphic curves. This proves the theorem. \square

Corollary 4.2.1. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. If there exists a positive constant δ such that*

$$\cos \alpha(\cdot, 0) \geq \delta > \frac{29(\lambda - 1)}{\sqrt{(48 - 24\lambda)^2 + (29\lambda - 29)^2}}$$

and has the longtime existence. Then the mean curvature flow converges to a holomorphic curve at infinity.

Proof. We just consider the case of $\lambda > 1$.

Since $\delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, we can choose $K = \frac{24(2-\lambda)}{\sqrt{\frac{1}{\delta^2}-1}(\lambda-1)} > 29$, then

$$\cos \alpha(\cdot, 0) \geq \frac{K(\lambda - 1)}{\sqrt{(48 - 24\lambda)^2 + (K\lambda - K)^2}} > \frac{29(\lambda - 1)}{\sqrt{(48 - 24\lambda)^2 + (29\lambda - 29)^2}}.$$

Then we can obtain the result from the proof of Theorem 4.2. \square

5. PINCHING ESTIMATE

This section, we prove a theorem generalizing Theorem 3.2 in [HLY]. First we recall a lemma in [HLY]

Lemma 5.1. *For any $\eta > 0$ we have the inequality*

$$(42) \quad |\nabla A|^2 \geq \left(\frac{3}{4} - \eta\right)|\nabla H|^2 - \left(\frac{1}{4}\eta^{-1} - 1\right)|w|^2,$$

where $w_i^\alpha = \Sigma_l R_{\alpha l i l}$, $|w^\alpha|^2 = \Sigma_i |w_i^\alpha|^2$ and $|w|^2 = \Sigma_\alpha |w^\alpha|^2$.

Now we can prove the following theorem.

Theorem 5.1. *Suppose Σ is a symplectic surface in the Kähler surface (M, J, ω, \bar{g}) with positive holomorphic sectional curvature and $1 \leq \lambda < 1 + \frac{1}{200}$, and $|\bar{\nabla} Rm| \leq Kk_1(\lambda - 1)$ for a positive constant $K \leq \min\{2, 2k_1\}$. For any σ satisfying $\frac{1}{2} + \frac{24(\lambda-1)}{1-34(\lambda-1)} < \sigma \leq \frac{2}{3}$, and we set*

$$b = \frac{2\sigma - 1}{\sigma}(8 - 7\lambda) - 4K(\lambda - 1),$$

and

$$\begin{aligned} a_1 &= 9(\lambda + 1)^2, \\ a_2 &= 9(\lambda + 1)^2 - \frac{12(3 - 4\sigma)}{2\sigma - 1}b, \\ a_3 &= \frac{350 - 444\sigma}{2\sigma - 1}(\lambda - 1) + \frac{8(3 - 4\sigma)}{2\sigma - 1}\left(23\lambda - \frac{41}{2}\right)b - \frac{8(3 - 4\sigma)(\sigma + 1)}{(2\sigma - 1)^2}b^2. \\ t_0 &= \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_1}, \quad \delta = \max\left\{t_0, \frac{13\lambda - 10}{3(\lambda + 2)}\right\} \end{aligned}$$

If we assume that $|A|^2 \leq \sigma|H|^2 + bk_1$ and $\cos \alpha \geq \sqrt{\delta}$ holds on the initial surface, then it remains true along the symplectic mean curvature flow.

Proof. Since we assume $\lambda \leq 1 + \frac{1}{200}$, we have

$$\frac{29(\lambda - 1)}{\sqrt{(48 - 24\lambda)^2 + (29\lambda - 29)^2}} < \frac{\sqrt{3}}{3} \leq \sqrt{\frac{13\lambda - 10}{3(\lambda + 2)}} \leq \sqrt{\delta}.$$

Then by Theorem 4.1 we know $\cos \alpha \geq \sqrt{\delta}$ is preserved by the mean curvature flow.

Since $|\nabla Rm| \leq Kk_1(\lambda - 1)$, Then

$$\begin{aligned} \frac{\partial}{\partial t}|A|^2 &= \Delta|A|^2 + 16Kk_1(\lambda - 1)|A| - 2|\nabla A|^2 - 4R_{lijk}h_{lk}^\alpha h_{ij}^\alpha + 8R_{\alpha\beta jk}h_{ik}^\beta h_{ij}^\alpha \\ &\quad - 4R_{likk}h_{lj}^\alpha h_{ij}^\alpha + 2R_{\alpha k\beta k}h_{ij}^\beta h_{ij}^\alpha + 2P_1 + 2P_2 \\ &\leq \Delta|A|^2 - 2|\nabla A|^2 - 4R_{lijk}h_{lk}^\alpha h_{ij}^\alpha + 8R_{\alpha\beta jk}h_{ik}^\beta h_{ij}^\alpha \\ &\quad - 4R_{likk}h_{lj}^\alpha h_{ij}^\alpha + 2R_{\alpha k\beta k}h_{ij}^\beta h_{ij}^\alpha + 2P_1 + 2P_2 + 8Kk_1(\lambda - 1) + 8Kk_1(\lambda - 1)|A|^2 \end{aligned}$$

From the calculation in the proof of Theorem 3.2 in [HLY], we have

- $-4R_{lijk}h_{lk}^\alpha h_{ij}^\alpha = -4R_{1212}(|A|^2 - |H|^2)$,
- $8R_{\alpha\beta jk}h_{ik}^\beta h_{ij}^\alpha = 8R_{1234}(|A|^2 - |\nabla J_{\Sigma_t}|^2)$,
- $-4R_{likk}h_{lj}^\alpha h_{ij}^\alpha = -R_{1212}|A|^2$,
- $2R_{\alpha k\beta k}h_{ij}^\beta h_{ij}^\alpha = 2R_{33}(h_{ij}^3)^2 + 2R_{44}(h_{ij}^4)^2 + 4R_{34}h_{ij}^3 h_{ij}^4 - 2R_{3434}|A|^2$.

Hence we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)|A|^2 &\leq -2|\nabla A|^2 + 8(R_{1234} - R_{1212})|A|^2 - 2R_{3434}|A|^2 + 4R_{1212}|H|^2 \\ &\quad - 8R_{1234}|\nabla J_{\Sigma_t}|^2 + 2R_{33}(h_{ij}^3)^2 + 2R_{44}(h_{ij}^4)^2 + 4R_{34}h_{ij}^3 h_{ij}^4 \\ &\quad + 2P_1 + 2P_2 + 8Kk_1(\lambda - 1) + 8Kk_1(\lambda - 1)|A|^2 \end{aligned}$$

Since

$$2R_{\alpha k\beta k}H^\alpha H^\beta = 2R_{33}(H^3)^2 + 2R_{44}(H^4)^2 + 4R_{34}H^3 H^4 - 2R_{3434}|H|^2,$$

by (5), we have

$$(43) \quad \left(\frac{\partial}{\partial t} - \Delta\right)|H|^2 = -2|\nabla H|^2 + 2R_{33}(H^3)^2 + 2R_{44}(H^4)^2 + 4R_{34}H^3 H^4 - 2R_{3434}|H|^2 + 2P_3$$

Hence

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right)(|A|^2 - \sigma|H|^2) \\ &\leq -2(|\nabla A|^2 - \sigma|\nabla H|^2) + 8(R_{1234} - R_{1212})(|A|^2 - \sigma|H|^2) + [8\sigma R_{1234} + (4 - 8\sigma)R_{1212}]\sigma|H|^2 \\ (44) \quad &- 8R_{1234}|\nabla J_{\Sigma_t}|^2 + 2R_{33}(|A|^2 - \sigma|H|^2) + 2(R_{44} - R_{33})(|h_4|^2 - \sigma(H^4)^2) \\ &+ 4R_{34}(h_{ij}^3 h_{ij}^4 - \sigma H^3 H^4) - 2R_{3434}(|A|^2 - \sigma|H|^2) \\ &+ 2P_1 + 2P_2 - 2\sigma P_3 + 8Kk_1(\lambda - 1) + 8Kk_1(\lambda - 1)\sigma|H|^2, \end{aligned}$$

where $|h_\alpha|^2$ and H^α are defined by

$$|h_\alpha|^2 = h_{ij}^\alpha h_{ij}^\alpha, \quad H^\alpha = \sum_i h_{ii}^\alpha.$$

Let $Q = |A|^2 - \sigma|H|^2 - bk_1$. By assumption, we have $\cos^2 \alpha \geq \frac{13\lambda-10}{3(\lambda+2)}$, then by Lemma 2.4, we have $R_{1234} \geq 0$. Since $|\bar{\nabla} J_{\Sigma_t}|^2 \geq \frac{1}{2}|H|^2$, we have

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} - \Delta\right)Q \\
 &= \left(\frac{\partial}{\partial t} - \Delta\right)(|A|^2 - \sigma|H|^2) \\
 &\leq -2(|\nabla A|^2 - \sigma|\nabla H|^2) + 8(R_{1234} - R_{1212})(Q + bk_1) + [8\sigma R_{1234} + (4 - 8\sigma)R_{1212}]|H|^2 \\
 &\quad - 8R_{1234}|\bar{\nabla} J_{\Sigma_t}|^2 + 2R_{33}(Q + bk_1) + 2(R_{44} - R_{33})(|h_4|^2 - \sigma(H^4)^2) + 4R_{34}(h_{ij}^3 h_{ij}^4 - \sigma H^3 H^4) \\
 &\quad - 2R_{3434}(Q + bk_1) + 2P_1 + 2P_2 - 2\sigma P_3 + 8Kk_1(\lambda - 1) + 8Kk_1(\lambda - 1)(Q + \sigma|H|^2 + bk_1) \\
 &\stackrel{(45)}{\leq} -2(|\nabla A|^2 - \sigma|\nabla H|^2) + 8(R_{1234} - R_{1212})(Q + bk_1) + [8\sigma R_{1234} + (4 - 8\sigma)R_{1212}]|H|^2 \\
 &\quad - 4R_{1234}|H|^2 + 2R_{33}(Q + bk_1) + 2(R_{44} - R_{33})(|h_4|^2 - \sigma(H^4)^2) + 4R_{34}(h_{ij}^3 h_{ij}^4 - \sigma H^3 H^4) \\
 &\quad - 2R_{3434}(Q + bk_1) + 2P_1 + 2P_2 - 2\sigma P_3 + 8Kk_1(\lambda - 1) + 8Kk_1(\lambda - 1)(Q + \sigma|H|^2 + bk_1) \\
 &= -2(|\nabla A|^2 - \sigma|\nabla H|^2) + CQ + [(8\sigma - 4)(R_{1234} - R_{1212}) + 8\sigma Kk_1(\lambda - 1)]|H|^2 \\
 &\quad + 2(R_{44} - R_{33})(|h_4|^2 - \sigma(H^4)^2) + 4R_{34}(h_{ij}^3 h_{ij}^4 - \sigma H^3 H^4) + 2P_1 + 2P_2 - 2\sigma P_3 \\
 &\quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1 + 8Kk_1(\lambda - 1),
 \end{aligned}$$

here C is a function. In the following, C always means a function, which might be different at places.

In Lemma 5.1, we choose $\eta = \frac{3}{4} - \sigma$, then

$$|\nabla A|^2 \geq \sigma|\nabla H|^2 - \frac{4\sigma - 2}{3 - 4\sigma}|w|^2,$$

where

$$|w|^2 = R_{3212}^2 + R_{3121}^2 + R_{4121}^2 + R_{4212}^2.$$

Assume $\alpha \in [0, \frac{\pi}{2})$ and $y \geq 0, z \geq 0$. Since $y^2 + z^2 = \sin^2 \alpha$, then by Lemma 2.4, we have

$$\begin{aligned}
 & |w|^2 \\
 &\leq \frac{k_1^2}{512} \{ [(23 + 6(\cos^2 \alpha + z^2))^2 + (23 + 6(\cos^2 \alpha + y^2))^2(\lambda - 1)^2 \\
 &\quad + 12 \cos \alpha [(23 + 6(\cos^2 \alpha + z^2))y + (23 + 6(\cos^2 \alpha + y^2))z](\lambda^2 - 1) \\
 &\quad + 144 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2] \} \\
 &\leq \frac{k_1^2}{512} [1682(\lambda - 1)^2 + 261(\lambda^2 - 1) + 144 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2],
 \end{aligned}$$

where we have used Cauchy inequality,

$$\begin{aligned}
 & [(23 + 6(\cos^2 \alpha + z^2))y + (23 + 6(\cos^2 \alpha + y^2))z]^2 \\
 &\leq [(23 + 6(\cos^2 \alpha + z^2))^2 + (23 + 6(\cos^2 \alpha + y^2))^2](y^2 + z^2) \\
 &\leq 2 \cdot 29^2 \sin^2 \alpha.
 \end{aligned}$$

Since $1 \leq \lambda \leq 1 + \frac{1}{100}$, we have

$$|w|^2 \leq \frac{k_1^2}{32} [34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2].$$

Since $\frac{1}{2} < \sigma < \frac{3}{4}$, we have

$$(46) \quad |\nabla A|^2 \geq \sigma |\nabla H|^2 - \frac{(2\sigma - 1)k_1^2}{16(3 - 4\sigma)} [34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2].$$

Hence we obtain

$$(47) \quad \begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) Q \\ & \leq CQ + [(8\sigma - 4)(R_{1234} - R_{1212}) + 8\sigma K k_1(\lambda - 1)] |H|^2 \\ & \quad + 2(R_{44} - R_{33})(|h_4|^2 - \sigma(H^4)^2) + 4R_{34}(h_{ij}^3 h_{ij}^4 - \sigma H^3 H^4) + 2P_1 + 2P_2 - 2\sigma P_3 \\ & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8K k_1(\lambda - 1)] b k_1 + 8K k_1(\lambda - 1) \\ & \quad + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)} [34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2]. \end{aligned}$$

At the point $|H| \neq 0$, we choose $\{e_3, e_4\}$ for $N\Sigma$ such that $e_3 = H/|H|$ and choose $\{e_1, e_2\}$ for $T\Sigma$ such that $h_{ij}^3 = \lambda_i \delta_{ij}$. Set $h_{ij}^\alpha = \mathring{h}_{ij}^\alpha + \frac{1}{2} H^\alpha g_{ij}$, then $\mathring{h}_{ij}^4 = h_{ij}^4, \mathring{h}_{ij}^3 = h_{ij}^3 - \frac{1}{2} |H| g_{ij}$. Denote the norm of $(h_{ij}^\alpha), (\mathring{h}_{ij}^\alpha)$ by $|h_\alpha|, |\mathring{h}_\alpha|$ respectively. Then $H^3 = |H|$ and $H^4 = 0$. From the calculation from the proof of Theorem 3.2 in [HLY], we have

$$\begin{aligned} & 2P_1 + 2P_2 - 2\sigma P_3 \\ & \leq 2|\mathring{h}_3|^4 + 2|\mathring{h}_4|^4 + (2 - 2\sigma)|\mathring{h}_3|^2 |H|^2 - \frac{2\sigma - 1}{2} |H|^4 + 8|\mathring{h}_3|^2 |\mathring{h}_4|^2. \end{aligned}$$

Then we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) Q \\ & \leq CQ + [(8\sigma - 4)(R_{1234} - R_{1212}) + 8\sigma K k_1(\lambda - 1)] |H|^2 \\ & \quad + 2(R_{44} - R_{33})|\mathring{h}_4|^2 + 4R_{34}\mathring{h}_{ij}^3 \mathring{h}_{ij}^4 \\ & \quad + 2|\mathring{h}_3|^4 + 2|\mathring{h}_4|^4 + (2 - 2\sigma)|\mathring{h}_3|^2 |H|^2 - \frac{2\sigma - 1}{2} |H|^4 + 8|\mathring{h}_3|^2 |\mathring{h}_4|^2 \\ & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8K k_1(\lambda - 1)] b k_1 + 8K k_1(\lambda - 1) \\ & \quad + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)} [34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha (\lambda + 1)^2] \end{aligned}$$

Since $|H|^2 = \frac{2}{2\sigma-1}(|\dot{h}_3|^2 + |\dot{h}_4|^2 - Q - bk_1)$, then

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} - \Delta\right)Q \\
 & \leq CQ + \frac{2}{2\sigma-1}[(8\sigma-4)(R_{1234} - R_{1212}) + 8\sigma Kk_1(\lambda-1)](|\dot{h}_3|^2 + |\dot{h}_4|^2) \\
 & \quad + 2(R_{44} - R_{33})|\dot{h}_4|^2 + 2|R_{34}|(|\dot{h}_3|^2 + |\dot{h}_4|^2) + 2|\dot{h}_3|^4 + 2|\dot{h}_4|^4 \\
 & \quad + \frac{4-4\sigma}{2\sigma-1}|\dot{h}_3|^2(|\dot{h}_3|^2 + |\dot{h}_4|^2 - bk_1) - \frac{2}{2\sigma-1}(|\dot{h}_3|^2 + |\dot{h}_4|^2 - bk_1)^2 + 8|\dot{h}_3|^2|\dot{h}_4|^2 \\
 & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda-1)]bk_1 + 8Kk_1(\lambda-1) \\
 & \quad + \frac{(2\sigma-1)k_1^2}{8(3-4\sigma)}[34(\lambda-1) + 9\sin^2\alpha\cos^2\alpha(\lambda+1)^2] \\
 & \quad - \frac{2}{2\sigma-1}[(8\sigma-4)(R_{1234} - R_{1212}) + 8\sigma Kk_1(\lambda-1)]bk_1 \\
 & \leq CQ + \frac{4\sigma-4}{2\sigma-1}|\dot{h}_4|^4 + \frac{12\sigma-8}{2\sigma-1}|\dot{h}_3|^2|\dot{h}_4|^2 + \frac{4\sigma-4}{2\sigma-1}bk_1|\dot{h}_3|^2 \\
 & \quad + [8(R_{1234} - R_{1212}) + 2|R_{34}| + 2|R_{44} - R_{33}| + \frac{16\sigma Kk_1(\lambda-1)}{2\sigma-1} + \frac{4bk_1}{2\sigma-1}](|\dot{h}_3|^2 + |\dot{h}_4|^2) \\
 & \quad + [2R_{33} - 2R_{3434} + 8Kk_1(\lambda-1) - \frac{16\sigma Kk_1}{2\sigma-1}(\lambda-1)]bk_1 + 8Kk_1(\lambda-1) \\
 & \quad + \frac{(2\sigma-1)k_1^2}{8(3-4\sigma)}[34(\lambda-1) + 9\sin^2\alpha\cos^2\alpha(\lambda+1)^2] - \frac{2b^2k_1^2}{2\sigma-1} \\
 & = CQ + \frac{4\sigma-4}{2\sigma-1}(|\dot{h}_4|^2 - \frac{bk_1}{2})^2 + \frac{12\sigma-8}{2\sigma-1}|\dot{h}_3|^2|\dot{h}_4|^2 \\
 & \quad + [8(R_{1234} - R_{1212}) + 2|R_{34}| + 2|R_{44} - R_{33}| + \frac{16\sigma Kk_1(\lambda-1)}{2\sigma-1} + \frac{4\sigma bk_1}{2\sigma-1}](|\dot{h}_3|^2 + |\dot{h}_4|^2) \\
 & \quad + [2R_{33} - 2R_{3434} + 8Kk_1(\lambda-1) - \frac{16\sigma Kk_1}{2\sigma-1}(\lambda-1)]bk_1 + 8Kk_1(\lambda-1) \\
 & \quad + \frac{(2\sigma-1)k_1^2}{8(3-4\sigma)}[34(\lambda-1) + 9\sin^2\alpha\cos^2\alpha(\lambda+1)^2] - \frac{\sigma+1}{2\sigma-1}b^2k_1^2
 \end{aligned}$$

Set

$$C_1 = 8(R_{1234} - R_{1212}) + 2|R_{34}| + 2|R_{44} - R_{33}| + \frac{16\sigma Kk_1(\lambda-1)}{2\sigma-1} + \frac{4\sigma bk_1}{2\sigma-1},$$

and

$$\begin{aligned}
 C_2 = & [2R_{33} - 2R_{3434} + 8Kk_1(\lambda-1) - \frac{16\sigma Kk_1}{2\sigma-1}(\lambda-1)]bk_1 + 8Kk_1(\lambda-1) \\
 & + \frac{(2\sigma-1)k_1^2}{8(3-4\sigma)}[34(\lambda-1) + 9\sin^2\alpha\cos^2\alpha(\lambda+1)^2] - \frac{\sigma+1}{2\sigma-1}b^2k_1^2.
 \end{aligned}$$

Then we have

$$\begin{aligned}
 (48) \quad \left(\frac{\partial}{\partial t} - \Delta\right)Q \leq & CQ + \frac{4\sigma-4}{2\sigma-1}(|\dot{h}_4|^2 - \frac{bk_1}{2})^2 + \frac{12\sigma-8}{2\sigma-1}|\dot{h}_3|^2|\dot{h}_4|^2 \\
 & + C_1(|\dot{h}_3|^2 + |\dot{h}_4|^2) + C_2.
 \end{aligned}$$

By Lemma 2.4, we can estimate C_1 as following

$$\begin{aligned}
 C_1 &\leq \frac{2}{3}[(10 + 6 \cos^2 \alpha)\lambda - (10 + 3 \sin^2 \alpha)]k_1 - 2[(3 + 3 \cos^2 \alpha) - 2\lambda]k_1 \\
 &\quad + \frac{(29 - 6 \cos^2 \alpha)(\lambda - 1)}{8}k_1 + 9(\lambda - 1)k_1 + \frac{16\sigma K k_1(\lambda - 1)}{2\sigma - 1} + \frac{4\sigma b k_1}{2\sigma - 1} \\
 &= [\frac{559 + 78 \cos^2 \alpha}{24}\lambda - \frac{655 + 78 \cos^2 \alpha}{24} + \frac{16K(\lambda - 1)}{2\sigma - 1} + \frac{4\sigma b}{2\sigma - 1}]k_1 \\
 &\leq [32(\lambda - 1) - 4\lambda + \frac{16\sigma K(\lambda - 1)}{2\sigma - 1} + \frac{4\sigma b}{2\sigma - 1}]k_1
 \end{aligned}$$

Then by the condition $b = \frac{2\sigma-1}{\sigma}(8 - 7\lambda) - 4K(\lambda - 1)$, we know $C_1 \leq 0$.

By Lemma 2.4 and since $K = \min\{2, 2k_1\}$, we can estimate C_2 as following

$$\begin{aligned}
 C_2 &\leq [7\lambda - \frac{9 + 3 \cos^2 \alpha}{2} + 16(\lambda - 1) - \frac{16K\sigma(\lambda - 1)}{2\sigma - 1}]bk_1^2 \\
 &\quad + 16k_1^2(\lambda - 1) + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha(\lambda + 1)^2] - \frac{\sigma + 1}{2\sigma - 1}b^2k_1^2 \\
 &= \{[23\lambda - \frac{41 + 3 \cos^2 \alpha}{2} - \frac{16K\sigma(\lambda - 1)}{2\sigma - 1}]b - \frac{\sigma + 1}{2\sigma - 1}b^2 \\
 &\quad + 16(\lambda - 1) + \frac{(2\sigma - 1)}{8(3 - 4\sigma)}[34(\lambda - 1) + 9 \sin^2 \alpha \cos^2 \alpha(\lambda + 1)^2]\}k_1^2
 \end{aligned}$$

Set $t = \cos^2 \alpha$. Then

$$\begin{aligned}
 C_2 &\leq \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}\{-9(\lambda + 1)^2t^2 + [9(\lambda + 1)^2 - \frac{12(3 - 4\sigma)}{2\sigma - 1}b]t \\
 &\quad + \frac{350 - 444\sigma}{2\sigma - 1}(\lambda - 1) + \frac{8(3 - 4\sigma)}{2\sigma - 1}(23\lambda - \frac{41}{2})b - \frac{8(3 - 4\sigma)(\sigma + 1)}{(2\sigma - 1)^2}b^2\}
 \end{aligned}$$

and

$$(49) \quad f(t) = -a_1t^2 + a_2t + a_3,$$

where

$$\begin{aligned}
 a_1 &= 9(\lambda + 1)^2, \\
 a_2 &= 9(\lambda + 1)^2 - \frac{12(3 - 4\sigma)}{2\sigma - 1}b, \\
 a_3 &= \frac{350 - 444\sigma}{2\sigma - 1}(\lambda - 1) + \frac{8(3 - 4\sigma)}{2\sigma - 1}(23\lambda - \frac{41}{2})b - \frac{8(3 - 4\sigma)(\sigma + 1)}{(2\sigma - 1)^2}b^2. \\
 t_0 &= \frac{a_2 + \sqrt{a_2^2 + 4a_1a_3}}{2a_1}.
 \end{aligned}$$

Then $C_2 \leq \frac{(2\sigma-1)k_1^2}{8(3-4\sigma)}f(t)$ and $f(t_0) = 0$.

We first to check that $f(1) < 0$, i.e., $-a_1 + a_2 + a_3 < 0$, then implies that $t_0 < 1$. That is

$$\frac{350 - 444\sigma}{2\sigma - 1}(\lambda - 1) + \frac{8(3 - 4\sigma)}{2\sigma - 1}(23\lambda - 22)b - \frac{8(3 - 4\sigma)(\sigma + 1)}{(2\sigma - 1)^2}b^2 < 0.$$

Set $x = \lambda - 1$. Then $b = \frac{2\sigma-1}{\sigma}(1-7x) - 4Kx$, we have

$$\begin{aligned}
 f(1) &= \frac{350-444\sigma}{2\sigma-1}x + \frac{8(3-4\sigma)}{2\sigma-1}(23x+1)\left(\frac{2\sigma-1}{\sigma}(1-7x) - 4Kx\right) \\
 &\quad - \frac{8(3-4\sigma)(\sigma+1)}{(2\sigma-1)^2}\left(\frac{2\sigma-1}{\sigma}(1-7x) - 4Kx\right)^2 \\
 &= \frac{350-444\sigma}{2\sigma-1}x + \frac{8(3-4\sigma)}{\sigma}(23x+1)(1-7x) - \frac{32(3-4\sigma)Kx}{2\sigma-1} \\
 &\quad - \frac{8(3-4\sigma)(\sigma+1)}{\sigma^2}\left(1 - \left(7 + \frac{4\sigma K}{2\sigma-1}\right)x\right)^2 \\
 &\leq \frac{350-444\sigma}{2\sigma-1}x + \frac{8(3-4\sigma)}{\sigma}(23x+1)(1-7x) - \frac{32(3-4\sigma)}{2\sigma-1}Kx \\
 &\quad - \frac{8(3-4\sigma)(\sigma+1)}{\sigma^2} + \frac{16(3-4\sigma)(\sigma+1)}{\sigma^2}\left(7 + \frac{4\sigma K}{2\sigma-1}\right)x \\
 &\leq \frac{350-444\sigma}{2\sigma-1}x + \frac{8(3-4\sigma)}{\sigma}(1+16x) - \frac{8(3-4\sigma)(\sigma+1)}{\sigma^2} \\
 &\quad + \frac{112(3-4\sigma)(\sigma+1)x}{\sigma^2} + \frac{64(3-4\sigma)Kx}{\sigma(2\sigma-1)} \\
 &\leq \frac{350-444\sigma}{2\sigma-1}x + \frac{8(3-4\sigma)}{\sigma}(1+16x) - \frac{8(3-4\sigma)(\sigma+1)}{\sigma^2} \\
 &\quad + \frac{112(3-4\sigma)(\sigma+1)x}{\sigma^2} + \frac{128(3-4\sigma)x}{\sigma(2\sigma-1)} \\
 &\leq \frac{8(3-4\sigma)}{\sigma^2}(-1 + (30\sigma+14)x) + (222 - 956\sigma + \frac{384}{\sigma})\frac{x}{2\sigma-1} \\
 &\leq \frac{2}{\sigma^2}[4(3-4\sigma)(-1 + (30\sigma+14)x) + (-478\sigma^3 + 111\sigma^2 + 192\sigma)\frac{x}{2\sigma-1}] \\
 &\leq \frac{2}{\sigma^2}\left(\frac{8}{3}(-1 + 34x) + 128\frac{x}{2\sigma-1}\right) \\
 &= \frac{16}{3\sigma^2}(-1 + 34x + 48\frac{x}{2\sigma-1}).
 \end{aligned}$$

Since $0 \leq x \leq \frac{1}{200}$ and $\frac{1}{2} + \frac{24x}{1-34x} \leq \frac{2}{3}$, we obtain that $-1 + 34x + 48\frac{x}{2\sigma-1} < 0$. So $f(1) < 0$. Then we have $f(t) \leq 0$ for $t_0 \leq t \leq 1$. Hence $C_2 \leq 0$ for $\cos \alpha \geq t_0$.

At the point $|H| = 0$, it is easy to obtain that (also see the inequality (3.12) in [HLY])

$$(50) \quad 2P_1 + 2P_2 \leq 3|A|^4,$$

and $P_3 = 0$. Thus, $|A|^2 = Q + bk_1$, by (38) we have

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} - \Delta\right)Q \\
 & \leq -CQ + 2(R_{44} - R_{33})|h_4|^2 + 4R_{34}h_{ij}^3h_{ij}^4 + 3|A|^4 \\
 & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1 + 8Kk_1(\lambda - 1) \\
 & \quad + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9\sin^2\alpha\cos^2\alpha(\lambda + 1)^2] \\
 & \leq -CQ + 2|R_{44} - R_{33}||A|^2 + 2|R_{34}||A|^2 + 3(Q + bk_1)^2 \\
 & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1 + 8Kk_1(\lambda - 1) \\
 & \quad + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9\sin^2\alpha\cos^2\alpha(\lambda + 1)^2] \\
 & = -CQ + 2|R_{44} - R_{33}||Q + bk_1| + 2|R_{34}||Q + bk_1| \\
 & \quad + [8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1 + 8Kk_1(\lambda - 1) \\
 & \quad + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9\sin^2\alpha\cos^2\alpha(\lambda + 1)^2] + 3b^2k_1^2 \\
 & = -CQ + \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9\sin^2\alpha\cos^2\alpha(\lambda + 1)^2] + 3b^2k_1^2 + 8Kk_1(\lambda - 1) \\
 & \quad + [2|R_{44} - R_{33}| + 2|R_{34}| + 8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1
 \end{aligned}$$

Set

$$\begin{aligned}
 \tilde{C}_2 &= \frac{(2\sigma - 1)k_1^2}{8(3 - 4\sigma)}[34(\lambda - 1) + 9\sin^2\alpha\cos^2\alpha(\lambda + 1)^2] + 3b^2k_1^2 + 8Kk_1(\lambda - 1) \\
 & \quad + [2|R_{44} - R_{33}| + 2|R_{34}| + 8(R_{1234} - R_{1212}) + 2R_{33} - 2R_{3434} + 8Kk_1(\lambda - 1)]bk_1.
 \end{aligned}$$

Then it is easy to get

$$\tilde{C}_2 = C_1bk_1 + C_2 + \frac{3\sigma - 2}{2\sigma - 1}b^2k_1^2.$$

Since $\frac{1}{2} < \sigma \leq \frac{2}{3}$, $C_1 \leq 0$ and $C_2 \leq 0$, we have $\tilde{C}_2 \leq 0$.

Therefore, from the above argument, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)Q \leq CQ.$$

for some function C . Applying the maximum principle for the above parabolic equation, we know that

$$Q \leq 0$$

along the flow, if it is true on initial surface. \square

6. LONG TIME EXISTENCE AND CONVERGENCE

In this section we prove the long time existence and convergence of the symplectic mean curvature flow under the assumption of Theorem 1.4, using the same argument in [HLY]. For convenience of readers, we provide the detailed proof here.

Theorem 6.1. *Under the assumption of Theorem 1.4, the symplectic mean curvature flow exists for long time.*

Proof. Suppose f is a positive increasing function which will be determined later. Now we compute the evolution equation of $|H|^2 f(\frac{1}{\cos \alpha})$.

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)(|H|^2 f(\frac{1}{\cos \alpha})) \\ &= \left(\frac{\partial}{\partial t} - \Delta\right)|H|^2 f(\frac{1}{\cos \alpha}) + |H|^2 \left(\frac{\partial}{\partial t} - \Delta\right)f(\frac{1}{\cos \alpha}) - 2\nabla|H|^2 \cdot \nabla f(\frac{1}{\cos \alpha}). \end{aligned}$$

Under the assumptions of the theorem, we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha &= |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha + Ric(Je_1, e_2) \sin^2 \alpha \\ &\geq |\bar{\nabla} J_{\Sigma_t}|^2 \cos \alpha. \end{aligned}$$

By the evolution equation of $|H|^2$, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)|H|^2 \\ &= -2|\nabla H|^2 + 2R_{33}(H^3)^2 + 2R_{44}(H^4)^2 + 4R_{34}H^3H^4 - 2R_{3434}|H|^2 + 2P_3 \\ &\leq -2|\nabla H|^2 + (6k_2 - 3k_1)|H|^2 - \frac{1}{2}[(3 + 3\cos^2 \alpha)k_1 - 2k_2]|H|^2 \\ &\quad + \frac{1}{8}(23 + 6\sin^2 \alpha)(k_2 - k_1)|H|^2 + 2|H|^2|A|^2 \\ &= -2|\nabla H|^2 + [\frac{5}{8}(17\lambda - 13) - \frac{3}{4}(\lambda + 1)\cos^2 \alpha]k_1|H|^2 + 2|H|^2|A|^2. \end{aligned}$$

We have used $P_3 \leq |H|^2|A|^2$ in the above equation.

Denote $C = \frac{5}{8}(17\lambda - 13) - \frac{3}{4}(\lambda + 1)\delta$. Since $\cos \alpha \geq \sqrt{\delta} > 0$ and the pinching condition, we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)|H|^2 \\ &\leq -2|\nabla H|^2 + Ck_1|H|^2 + 2|H|^2(\sigma|H|^2 + bk_1) \\ &= -2|\nabla H|^2 + (C + 2b)k_1|H|^2 + 2\sigma|H|^4. \end{aligned}$$

Putting the above inequality into the evolution of $|H|^2 f(\frac{1}{\cos \alpha})$, we get that

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)(|H|^2 f(\frac{1}{\cos \alpha})) \\ &\leq f(\frac{1}{\cos \alpha})(-2|\nabla H|^2 + (C + 2b)k_1|H|^2 + 2\sigma|H|^4) \\ &\quad - |H|^2(f' \frac{|\bar{\nabla} J_{\Sigma_t}|^2}{\cos \alpha} + 2f' \frac{|\nabla \cos \alpha|^2}{\cos^3 \alpha} + f'' \frac{|\nabla \cos \alpha|^2}{\cos^4 \alpha}) \\ &\quad - 2 \frac{\nabla(f|H|^2) - |H|^2 \nabla f}{f} \cdot \nabla f(\frac{1}{\cos \alpha}) \\ (51) \quad &= (C + 2b)k_1 f|H|^2 + |H|^2 f(\frac{1}{\cos \alpha})(-2 \frac{|\nabla H|^2}{|H|^2} + 2\sigma|H|^2 - \frac{f'}{f} \frac{|\bar{\nabla} J_{\Sigma_t}|^2}{\cos \alpha}) \\ &\quad + |H|^2(-f'' + 2 \frac{(f')^2}{f} - 2f' \cos \alpha) \frac{|\nabla \cos \alpha|^2}{\cos^4 \alpha} \\ &\quad - 2|H|^2 \frac{\nabla(f|H|^2)}{f|H|^2} \cdot \nabla f(\frac{1}{\cos \alpha}) \end{aligned}$$

Set $\phi = f|H|^2$. At the point where $\phi \neq 0$, it is easy to see that

$$\nabla \phi = f \nabla |H|^2 + |H|^2 \nabla f = f \nabla |H|^2 - |H|^2 f' \frac{\nabla \cos \alpha}{\cos^2 \alpha},$$

i.e.,

$$(52) \quad \frac{\nabla \cos \alpha}{\cos^2 \alpha} = \frac{f}{f'} \left(\frac{\nabla |H|^2}{|H|^2} - \frac{\nabla \phi}{\phi} \right).$$

Plugging (52) into (51), we obtain

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta \right) \phi \\ & \leq (C + 2b)k_1\phi + \phi \left(-2 \frac{|\nabla H|^2}{|H|^2} + 2\sigma |H|^2 - \frac{f' \bar{\nabla} J_{\Sigma_t}|^2}{f \cos \alpha} \right) \\ & \quad + \frac{\phi f}{(f')^2} \left(-f'' + 2 \frac{(f')^2}{f} - 2f' \cos \alpha \right) \left(\frac{|\nabla |H|^2|^2}{|H|^4} - 2 \frac{\nabla |H|^2}{|H|^2} \cdot \frac{\nabla \phi}{\phi} + \frac{|\nabla \phi|^2}{\phi^2} \right) \\ & \quad + 2|H|^2 f' \frac{\nabla \phi}{\phi} \cdot \frac{\nabla \cos \alpha}{\cos^2 \alpha} \\ (53) \quad & \leq (C + 2b)k_1\phi + \phi \left(-\frac{f'}{2f \cos \alpha} |H|^2 + 2\sigma |H|^2 \right) \\ & \quad + \phi \left(-2 \frac{|\nabla H|^2}{|H|^2} - 4 \frac{f f''}{(f')^2} \frac{|\nabla |H|^2|^2}{|H|^2} + 8 \frac{|\nabla |H|^2|^2}{|H|^2} - 8 \frac{f |\nabla |H|| \cos \alpha}{f' |H|^2} \right) \\ & \quad + 2|H|^2 f' \frac{\nabla \phi}{\phi} \cdot \frac{\nabla \cos \alpha}{\cos^2 \alpha} + \phi \left(-\frac{f f''}{(f')^2} - 2 \frac{f}{f'} \cos \alpha + 2 \right) \left(\frac{|\nabla \phi|^2}{\phi^2} - 2 \frac{\nabla |H|^2}{|H|^2} \cdot \frac{\nabla \phi}{\phi} \right) \\ & = (C + 2b)k_1\phi + \phi \left(-\frac{f'}{2f \cos \alpha} + 2\sigma \right) |H|^2 + \phi \left(-4 \frac{f f''}{(f')^2} - 8 \frac{f \cos \alpha}{f'} + 6 \right) \frac{|\nabla |H||^2}{|H|^2} \\ & \quad + \phi \left(-\frac{f f''}{(f')^2} - 2 \frac{f}{f'} \cos \alpha + 2 \right) \left(\frac{|\nabla \phi|^2}{\phi^2} - 2 \frac{\nabla |H|^2}{|H|^2} \cdot \frac{\nabla \phi}{\phi} \right) + 2|H|^2 f' \frac{\nabla \phi}{\phi} \cdot \frac{\nabla \cos \alpha}{\cos^2 \alpha} \end{aligned}$$

Set $\frac{f}{f'} = g$, we choose g such that for $x \in [1, \frac{1}{\sqrt{\delta}}]$

$$\begin{cases} x/g \geq 4\sigma, \\ -4g' + 8g/x - 2 = 0. \end{cases}$$

We choose $c(x) = \frac{1}{2} - ax$ by solving the last equation, where a will be defined later. It reduces to solve the inequality

$$0 < \frac{1}{2} - ax \leq \frac{1}{4\sigma}, \quad x \in [1, \frac{1}{\sqrt{\delta}}],$$

i.e.,

$$\left(\frac{1}{2} - \frac{1}{4\sigma} \right) \frac{1}{x} \leq a < \frac{1}{2x}, \quad x \in [1, \frac{1}{\sqrt{\delta}}].$$

Note that

$$\cos^2 \alpha \geq \frac{13\lambda - 10}{3(\lambda + 2)} \geq \frac{1}{3} > \left(1 - \frac{1}{2\sigma} \right)^2$$

for any $\sigma \in (\frac{1}{2}, \frac{2}{3}]$.

Hence we can choose $a = \frac{1}{2} - \frac{1}{4\sigma}$, then

$$g = \frac{x}{2} - \left(\frac{1}{2} - \frac{1}{4\sigma}\right)x^2,$$

and

$$f(x) = \frac{(1-2a)^2 x^2}{(1-2ax)^2} = \frac{x^2}{(2\sigma - (2\sigma-1)x)^2}, \quad x \in [1, \frac{1}{\sqrt{\delta}}].$$

Then for any $x \in [1, \frac{1}{\sqrt{\delta}}]$,

$$1 \leq f(x) \leq \frac{1}{(2\sigma\sqrt{\delta} - (2\sigma-1))^2}.$$

By (44), we have

$$\begin{aligned} & \left(\frac{\partial}{\partial t} - \Delta\right)\phi \\ (54) \quad & \leq (C+2b)k_1\phi + \phi\left(-\frac{ff''}{(f')^2} - 2\frac{f}{f'}\cos\alpha + 2\right)\left(\frac{|\nabla\phi|^2}{\phi^2} - 2\frac{\nabla|H|^2}{|H|^2} \cdot \frac{\nabla\phi}{\phi}\right) \\ & \quad + 2|H|^2 f' \frac{\nabla\phi}{\phi} \cdot \frac{\nabla\cos\alpha}{\cos^2\alpha} \end{aligned}$$

This implies that

$$\frac{\partial}{\partial t}\phi_{max}(t) \leq (C+2b)k_1\phi_{max}(t),$$

where $\phi_{max}(t)$ mean the maximum of ϕ on Σ_t . Hence

$$|H|^2(t) \leq \phi_{max}(t) \leq e^{(C+2b)k_1 t} |H|^2(0) f\left(\frac{1}{\cos\alpha}\right)(0).$$

We have

$$|H|^2(t) \leq C_0 e^{C_1 t},$$

where C_0 depends only on $\max_{\Sigma_0} |H|^2$, λ and σ . Pinching inequality implies that

$$|A|^2 \leq C_2 e^{C_1 t} + b k_1.$$

We finish the proof of the theorem. \square

Theorem 6.2. *Under the assumption of Theorem 1.4, the symplectic mean curvature flow converges to a holomorphic curve at infinity.*

In fact, this theorem can follows from Corollary 4.2.1. But here we also provide the argument from [HLY].

Proof. Since $\cos^2\alpha \geq \frac{1}{3}$ and $\lambda \leq 1 + \frac{1}{200}$ and (32), we know $Ric(Je_1, e_2) > \frac{1}{3}k_1$. By the evolution equation of $\cos\alpha$

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\overline{\nabla}J_{\Sigma_t}|^2 \cos\alpha + Ric(Je_1, e_2) \sin^2\alpha,$$

we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha \geq \frac{1}{3}k_1 \sin^2\alpha.$$

Rewrite the above inequality as

$$\left(\frac{\partial}{\partial t} - \Delta\right)\sin^2(\alpha/2) \leq -\frac{4}{9}k_1 \sin^2(\alpha/2).$$

Applying the maximum principle, we get that $\sin^2(\alpha/2) \leq e^{-\frac{4}{9}k_1 t}$. By Theorem 6.1 we know that the symplectic mean curvature flow exists for long time. Thus for any $\epsilon > 0$, there exists $T > 0$ such that as $t > T$, we have

$$(55) \quad \begin{aligned} \cos \alpha &\geq 1 - \epsilon, \\ \sin \alpha &\leq 2\epsilon, \\ |\nabla \cos \alpha|^2 &\leq \sin^2 \alpha |\bar{\nabla} J_{\Sigma_t}|^2 \leq 2\epsilon |\bar{\nabla} J_{\Sigma_t}|^2 \leq 4\epsilon |A|^2. \end{aligned}$$

Therefore by pinching inequality,

$$(56) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \cos \alpha &\geq \frac{1}{2} |H|^2 \cos \alpha \geq \left(\frac{1}{2\sigma} |A|^2 - \frac{bk_1}{2\sigma}\right) \cos \alpha \\ &\geq \frac{1}{2\sigma} (1 - \epsilon) |A|^2 - \frac{bk_1}{2\sigma}. \end{aligned}$$

From the evolution equation of $|A|^2$, we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 \leq -2|\nabla A|^2 + C_1 |A|^4 + C_2 |A|^2 + C_3,$$

where C_1, C_2, C_3 are positive constants that depend only on k_1, λ, K .

Let $p > 1$ be a constant to be fixed later. For simplicity, we set $u = \cos \alpha$. Now we consider the function $\frac{|A|^2}{e^{pu}}$.

$$\begin{aligned} &\left(\frac{\partial}{\partial t} - \Delta\right) \frac{|A|^2}{e^{pu}} \\ &= 2\nabla \left(\frac{|A|^2}{e^{pu}}\right) \cdot \frac{\nabla e^{pu}}{e^{pu}} + \frac{1}{e^{2pu}} [e^{pu} \left(\frac{\partial}{\partial t} - \Delta\right) |A|^2 - |A|^2 \left(\frac{\partial}{\partial t} - \Delta\right) e^{pu}] \\ &\leq 2p \nabla \left(\frac{|A|^2}{e^{pu}}\right) \cdot \nabla u + \frac{1}{e^{2pu}} \{e^{pu} (C_1 |A|^4 + C_2 |A|^2 + C_3) - p |A|^2 e^{pu} [\frac{1}{2\sigma} (1 - \epsilon) |A|^2 - \frac{bk_1}{2\sigma} - p |\nabla u|^2]\}. \end{aligned}$$

Using (52) we obtain that,

$$\left(\frac{\partial}{\partial t} - \Delta\right) \frac{|A|^2}{e^{pu}} \leq 2p \nabla \left(\frac{|A|^2}{e^{pu}}\right) \cdot \nabla u + \frac{1}{e^{pu}} [(C_1 - \frac{1}{2\sigma} p(1 - \epsilon) + 4p^2 \epsilon) |A|^4 + C_4 |A|^2 + C_3]$$

Set $p^2 = \frac{1}{\epsilon}$, then

$$C_1 - \frac{1}{2\sigma} p(1 - \epsilon) + 4p^2 \epsilon = C_1 - \frac{1}{2\sigma} \epsilon^{-\frac{1}{2}} + \frac{1}{2\sigma} \epsilon^{\frac{1}{2}} + 4.$$

As t is sufficiently large, i.e., ϵ is sufficiently close to 0, we have

$$C_1 - \frac{1}{2\sigma} \epsilon^{-\frac{1}{2}} + \frac{1}{2\sigma} \epsilon^{\frac{1}{2}} + 4 \leq -1.$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right) \frac{|A|^2}{e^{pu}} \leq 2p \nabla \left(\frac{|A|^2}{e^{pu}}\right) \cdot \nabla u - \frac{|A|^4}{e^{2pu}} + \frac{C_4 |A|^2}{e^{pu}} + \frac{C_3}{e^{pu}}.$$

Applying the maximum principle, we conclude that $\frac{|A|^2}{e^{pu}}$ is uniformly bounded, thus $|A|^2$ is also uniformly bounded. Thus $F(\cdot, t)$ converges to F_∞ in C^2 as $t \rightarrow \infty$. Since $\sin^2(\frac{\alpha}{2}) \leq e^{-\frac{4}{9}k_1 t}$, we have $\cos \alpha \equiv 1$ at infinity. Thus the limiting surface F_∞ is a holomorphic curve. \square

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